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The 10th Anniversary Conference on  
Nonlinear Analysis and Convex Analysis  
July 4–8(9) at Chitose City Cultural Center Hokkaido, Japan

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Travel Information

## EXISTENCE THEOREM FOR A SECOND-ORDER IMPULSIVE FUNCTIONAL-DIFFERENTIAL EQUATION WITH A NONLOCAL CONDITION

HAYDAR AKÇA, VALÉRY COVACHEV, AND ZLATINKA COVACHEVA

**ABSTRACT.** An abstract second-order semilinear functional-differential equation such that the linear part of its right-hand side is given by the infinitesimal generator of a strongly continuous cosine family of bounded linear operators, and provided with impulse and nonlocal conditions is studied. Under not too restrictive conditions the existence of a mild solution is proved using Schauder's fixed point theorem.

### 1. INTRODUCTION

Many evolutionary processes in nature are characterized by the fact that at certain instants of time they undergo a rapid change of their states. The theory of the impulsive differential equations is one of the attractive branches of differential equations which has extensive realistic mathematical modelling applications in physics, chemistry, engineering, and biological and medical sciences. The nonlocal condition generalizes the classical initial condition. In our previous papers [1, 2] we found sufficient conditions for the existence, uniqueness and continuous dependence of a mild solution of a first-order impulsive functional-differential evolution nonlocal Cauchy problem such that the linear part of the right-hand side of the differential equation is given by the infinitesimal generator of a strongly continuous semigroup of bounded linear operators. In our recent paper [3] we proved theorems for existence and uniqueness of a mild and classical solution of an abstract second-order semilinear functional-differential equation such that the linear part of its right-hand side is given by the infinitesimal generator of a strongly continuous cosine family of bounded linear operators, and provided with impulse and nonlocal conditions.

### 2. PRELIMINARIES

In the present paper we consider the abstract second-order nonlinear impulsive functional-differential equation with nonlocal condition

$$(2.1) \quad \begin{aligned} x''(t) &= Ax(t) + f(t, x(t), x(b_1(t)), \dots, x(b_m(t)), \\ &\quad x'(t), x'(b_1(t)), \dots, x'(b_m(t))), \\ t &\in (0, T] \setminus \{\tau_1, \tau_2, \dots, \tau_\kappa\}, \\ \Delta x(\tau_k) &= I_k(x(\tau_k)), \\ \Delta x'(\tau_k) &= \bar{I}_k(x(\tau_k), x'(\tau_k)), \quad k = \overline{1, \kappa}, \end{aligned}$$

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$$\begin{aligned}x(0) &= x_0, \\x'(0) &= x_1 - g(x),\end{aligned}$$

where  $A$  is a linear operator from a real Banach space  $X$  with norm  $\|\cdot\|$  into itself,  $x : [0, T] \rightarrow X$ ,  $\Delta x(\tau_k) = x(\tau_k + 0) - x(\tau_k - 0) \equiv x(\tau_k + 0) - x(\tau_k)$ ,  $\Delta x'(\tau_k) = x'(\tau_k + 0) - x'(\tau_k - 0) \equiv x'(\tau_k + 0) - x'(\tau_k)$ ,  $0 < \tau_1 < \tau_2 < \dots < \tau_\kappa < T$  are the instants of impulse effect,  $f : [0, T] \times X^{2m+2} \rightarrow X$ ,  $b_i : [0, T] \rightarrow [0, T]$  ( $i = \overline{1, m}$ )  $I_k : X \rightarrow X$ ,  $\bar{I}_k : X^2 \rightarrow X$ ,  $x_0, x_1 \in X$ , and  $g(x)$  is a function with values in  $X$  to be specified later.

Next we shall need the following definitions [9, 6, 7, 8, 10].

**Definition 2.1.** A one-parameter family  $\{C(t) : t \in \mathbb{R}\}$  of bounded linear operators mapping the Banach space  $X$  into itself is called a *strongly continuous cosine family* if and only if

- 1)  $C(s+t) + C(s-t) = 2C(s)C(t)$  for all  $s, t \in \mathbb{R}$ .
- 2)  $C(0) = I$  (the identity operator).
- 3)  $C(t)x$  is continuous in  $t$  on  $\mathbb{R}$  for each fixed  $x \in X$ .

**Definition 2.2.** The *infinitesimal generator* of a strongly continuous cosine family  $\{C(t)\}$  is the operator  $A : X \supset \mathcal{D}(A) \rightarrow X$  defined by

$$Ax := \left. \frac{d^2}{dt^2} C(t)x \right|_{t=0}, \quad x \in \mathcal{D}(A),$$

where

$$\mathcal{D}(A) := \{x \in X : C(t)x \text{ is of class } C^2 \text{ with respect to } t\}.$$

**Example 2.3.** If  $A$  is a positive bounded operator in a Hilbert space  $X$ , then  $\{\cosh(\sqrt{A}t)\}$  is a strongly continuous cosine family of operators with infinitesimal generator  $A$ . If  $A$  is a negative bounded operator in a Hilbert space  $X$ , then  $\{\cos(\sqrt{-A}t)\}$  is a strongly continuous cosine family of operators with infinitesimal generator  $A$ .

We introduce the assumptions:

**A1:** The operator  $A$  is the infinitesimal generator of a strongly continuous cosine family  $\{C(t) : t \in \mathbb{R}\}$  of bounded linear operators from  $X$  to itself.

**A2:** The adjoint operator  $A^*$  is densely defined in  $X^*$ , i.e.,  $\overline{\mathcal{D}(A^*)} = X^*$ .

Let us denote

$$E := \{x \in X : C(t)x \text{ is of class } C^1 \text{ with respect to } t\}.$$

The associated sine family  $\{S(t) : t \in \mathbb{R}\}$  is defined by

$$S(t)x := \int_0^t C(s)x \, ds, \quad x \in X, \quad t \in \mathbb{R}.$$

Further on we denote by  $\|C(t)\|$ ,  $\|S(t)\|$  and  $\|A\|$  the operator norms of  $C(t)$ ,  $S(t)$  and  $A$  in the Banach space  $X$ , respectively. From Assumption **A1** it follows that there is a constant  $M \geq 1$  such that

$$(2.2) \quad \|C(t)\| \leq M \quad \text{and} \quad \|S(t)\| \leq M \quad \text{for} \quad t \in [0, T].$$

Following [5], we present a result obtained by J. Bochenek in [4].

Let us consider the Cauchy problem

$$(2.3) \quad \begin{aligned} x''(t) &= Ax(t) + h(t), & t \in (0, T], \\ x(0) &= x_0, \quad x'(0) = x_1. \end{aligned}$$

**Definition 2.4.** A function  $x : [0, T] \rightarrow X$  is said to be a *classical solution* of problem (2.3) if

$$\begin{aligned} x &\in C^1([0, T], X) \cap C^2((0, T], X), \\ x(0) &= x_0 \quad \text{and} \quad x'(0) = x_1, \\ x''(t) &= Ax(t) + h(t) \quad \text{for} \quad t \in (0, T]. \end{aligned}$$

**Theorem 2.5.** Suppose that

- 1) Assumptions **A1** and **A2** are satisfied;
- 2)  $h : [0, T] \rightarrow X$  is Lipschitz continuous;
- 3)  $x_0 \in \mathcal{D}(A)$  and  $x_1 \in E$ .

Then problem (2.3) has a unique classical solution given by the formula

$$x(t) = C(t)x_0 + S(t)x_1 + \int_0^t S(t-s)h(s) ds, \quad t \in [0, T].$$

It is easy to see that this result can be generalized for the impulsive system

$$(2.4) \quad x''(t) = Ax(t) + h(t), \quad t \in (0, T] \setminus \{\tau_1, \tau_2, \dots, \tau_\kappa\},$$

$$\Delta x(\tau_k) = I_k,$$

$$(2.5) \quad \Delta x'(\tau_k) = \bar{I}_k, \quad k = \overline{1, \kappa},$$

$$(2.6) \quad x(0) = x_0, \quad x'(0) = x_1.$$

For convenience we denote  $J = [0, T]$ ,  $J_0 = [0, \tau_1]$ ,  $J_k = (\tau_k, \tau_{k+1}]$ ,  $k = \overline{1, \kappa - 1}$ ,  $J_\kappa = (\tau_\kappa, T]$ ,  $J' = J \setminus \{0, \tau_1, \tau_2, \dots, \tau_\kappa\}$ . For a function  $x : J \rightarrow X$  we denote by  $x_k$  the restriction of  $x$  to  $J_k$ ,  $k = \overline{0, \kappa}$ , with  $\|x_k\|_{J_k} = \sup_{s \in J_k} \|x_k(s)\|$ . If the function  $x_k$

happens to be differentiable, we denote its derivative by  $x'_k$ . Further we define the following classes of functions:

$$\begin{aligned} PC(J, X) &= \{x : J \rightarrow X \mid x_k \in C(J_k, X), \quad k = \overline{0, \kappa}, \\ &\quad \text{and there exist } x(\tau_k + 0), \quad k = \overline{1, \kappa}\}, \\ PC^1(J, X) &= \{x \in PC(J, X) \mid x'_k \in C(J_k, X), \quad k = \overline{0, \kappa}, \\ &\quad \text{and there exist } x'(\tau_k + 0), \quad k = \overline{1, \kappa}\}. \end{aligned}$$

$PC(J, X)$  is a Banach space with norm  $\|x\|_{PC} = \max \{\|x_k\|_{J_k}, \quad k = \overline{0, \kappa}\}$ , and  $PC^1(J, X)$  is a Banach space with norm  $\|x\|_{PC^1} = \|x\|_{PC} + \|x'\|_{PC}$ .

**Definition 2.6.** A function  $x \in PC^1(J, X) \cap C^2(J', X)$  is called a *classical solution* of problem (2.4)–(2.6) if it satisfies the differential equation (2.4) on  $J'$  together with the impulse conditions (2.5) and the initial conditions (2.6).

**Theorem 2.7** ([3]). Suppose that

- 1) Assumptions **A1** and **A2** are satisfied;
- 2)  $h \in PC(J, X)$  is such that its restrictions to  $J_k$  are Lipschitz continuous,  $k = \overline{0, \kappa}$ ;

3)  $x_0 \in \mathcal{D}(A)$  and  $x_1 \in E$ .

4)  $I_k \in \mathcal{D}(A)$  and  $\bar{I}_k \in E$  for  $k = \overline{1, \kappa}$ .

Then problem (2.4)–(2.6) has a unique classical solution given by the formula

$$\begin{aligned} x(t) &= C(t)x_0 + S(t)x_1 + \int_0^t S(t-s)h(s) ds \\ &+ \sum_{0 < \tau_k < t} C(t-\tau_k)I_k + \sum_{0 < \tau_k < t} S(t-\tau_k)\bar{I}_k, \quad t \in J. \end{aligned}$$

Theorem 2.7 can be proved by applying Theorem 2.5 on each interval of continuity  $J_k$ ,  $k = \overline{0, \kappa}$ .

This theorem suggests the following definition.

**Definition 2.8.** A function  $x \in PC^1(J, X)$  satisfying the integro-summary equation

$$\begin{aligned} (2.7) \quad x(t) &= C(t)x_0 + S(t)(x_1 - g(x)) \\ &+ \int_0^t S(t-s)f(s, x(s), x(b_1(s)), \dots, x(b_m(s)), \\ &\quad x'(s), x'(b_1(s)), \dots, x'(b_m(s))) ds \\ &+ \sum_{0 < \tau_k < t} C(t-\tau_k)I_k(x(\tau_k)) \\ &+ \sum_{0 < \tau_k < t} S(t-\tau_k)\bar{I}_k(x(\tau_k), x'(\tau_k)), \quad t \in J, \end{aligned}$$

is said to be a *mild solution* of the nonlocal problem (2.1).

In the next section we present not too restrictive sufficient conditions for the nonlocal problem (2.1) to have a mild solution.

### 3. MAIN RESULT

For  $r > 0$  denote

$$X_r = \{u \in PC^1(J, X) : \|u\|_{PC^1} \leq r\}.$$

Next we introduce the assumption:

**A3:**  $x_0 \in E$ ,  $x_1 \in X$ ,  $f : J \times X^{2m+2} \rightarrow X$  is such that the function  $t \mapsto f(t, u_0, u_1, \dots, u_m, v_0, v_1, \dots, v_m)$  belongs to  $PC(J, X)$  for each  $(u_0, u_1, \dots, u_m, v_0, v_1, \dots, v_m) \in X^{2m+2}$  and the function  $(u_0, u_1, \dots, u_m, v_0, v_1, \dots, v_m) \mapsto f(t, u_0, u_1, \dots, u_m, v_0, v_1, \dots, v_m)$  belong to  $C(X^{2m+2}, X)$  for each  $t \in J$ ,  $b_i \in C(J, J)$  ( $i = \overline{1, m}$ ),  $I_k \in C(X, E)$ ,  $\bar{I}_k \in C(X^2, X)$ ,  $g \in C(PC^1(J, X), X)$  and there exist positive constant

$C_i$  ( $i = \overline{1, 4}$ ) and  $r$  such that

$$\begin{cases} \|f(t, u_0, u_1, \dots, u_m, v_0, v_1, \dots, v_m)\| \leq C_1 \text{ for} \\ t \in J, \|u_i\| + \|v_i\| \leq r \text{ } (i = \overline{0, m}), \\ \|g(u)\| \leq C_2 \text{ for } u \in X_r, \\ \|I_k(u)\| \leq C_3 \text{ for } \|u\| \leq r, k = \overline{1, \kappa}, \\ \|\bar{I}_k(u, v)\| \leq C_4 \text{ for } \|u\| + \|v\| \leq r, k = \overline{1, \kappa} \end{cases}$$

and

$$r \geq M \{(1 + \|A\|)\|x_0\| + 2\|x_1\| + 2TC_1 + 2C_2 + \kappa[(1 + \|A\|)C_3 + 2C_4]\},$$

where  $M$  is the constant from inequalities (2.2).

**Example 3.1.** Consider the scalar problem

$$\begin{aligned} (3.1) \quad x''(t) &= x(t) \\ &+ C_1 \left[ (x(t) + x'(t))^2/2 + \sum_{i=1}^m 2^{-i-1} (x(b_i(t)) + x'(b_i(t)))^2 \right], \\ &t \in (0, 2] \setminus \{\tau_1, \tau_2, \dots, \tau_\kappa\}, \\ \Delta x(\tau_k) &= c_k x(\tau_k), \quad k = \overline{1, \kappa}, \\ \Delta x'(\tau_k) &= c'_k (x(\tau_k) + x'(\tau_k)), \quad k = \overline{1, \kappa}, \\ x(0) &= x_0, \\ x'(0) &= x_1 - C_2 \sum_{j=1}^p 2^{-j} (x(t_j) + x'(t_j)), \end{aligned}$$

where  $0 < \tau_1 < \dots < \tau_\kappa < 2$ ,  $0 < t_1 < \dots < t_p \leq 2$  and

$$|c_k| \leq C_3, \quad |c'_k| \leq C_4 \quad \text{for } k = \overline{1, \kappa}.$$

Here  $X = \mathbb{R}$ ,  $A = 1$ ,  $C(t) = \cosh t$ ,  $S(t) = \sinh t$ ,  $\max_{0 \leq t \leq 2} \cosh t = \cosh 2 < 4$  since  $\operatorname{arcosh} 4 = 2.063437$ ,  $\max_{0 \leq t \leq 2} \sinh t = \sinh 2 < \cosh 2 < 4$ , thus we can choose  $M = 4$ .

If we take  $r = 1$ , it is easy to see that the constants  $C_i$ ,  $i = \overline{1, 4}$  in problem (3.1) can be chosen as the constants in condition **A3**. Then condition **A3** is satisfied if

$$\|x_0\| + \|x_1\| + 2C_1 + C_2 + \kappa(C_3 + C_4) \leq \frac{1}{8}.$$

Now we formulate our main result.

**Theorem 3.2.** Suppose that assumptions **A1** and **A3** are satisfied. Then problem (2.1) has a mild solution.

*Proof.* We can write equation (2.7) in an operator form

$$x = \mathcal{F}x,$$

where the operator  $\mathcal{F}$  is defined on the Banach space  $PC^1(J, X)$  by the formula

$$(3.2) \quad (\mathcal{F}u)(t) = C(t)x_0 + S(t)(x_1 - g(u))$$

$$\begin{aligned}
& + \int_0^t S(t-s) f(s, u(s), u(b_1(s)), \dots, u(b_m(s)), \\
& \quad u'(s), u'(b_1(s)), \dots, u'(b_m(s))) ds \\
& + \sum_{0 < \tau_k < t} C(t - \tau_k) I_k(u(\tau_k)) \\
& + \sum_{0 < \tau_k < t} S(t - \tau_k) \bar{I}_k(u(\tau_k), u'(\tau_k)), \quad t \in J.
\end{aligned}$$

It is easy to see that  $\mathcal{F}u \in PC(J, X)$  for  $u \in PC^1(J, X)$ . Moreover, since

$$\begin{aligned}
(3.3) \quad \frac{d}{dt}(\mathcal{F}u)(t) & = AS(t)x_0 + C(t)(x_1 - g(u)) \\
& + \int_0^t C(t-s) f(s, u(s), u(b_1(s)), \dots, u(b_m(s)), \\
& \quad u'(s), u'(b_1(s)), \dots, u'(b_m(s))) ds \\
& + \sum_{0 < \tau_k < t} AS(t - \tau_k) I_k(u(\tau_k)) \\
& + \sum_{0 < \tau_k < t} C(t - \tau_k) \bar{I}_k(u(\tau_k), u'(\tau_k)), \quad t \in J,
\end{aligned}$$

we have  $\frac{d}{dt}\mathcal{F}u \in PC(J, X)$  and, consequently,  $\mathcal{F}u \in PC^1(J, X)$ . Thus the operator  $\mathcal{F}$  maps the Banach space  $PC^1(J, X)$  into itself.

In order to find a mild solution  $x$  of problem (2.1) we are looking for a fixed point of the operator  $\mathcal{F}$  in a suitably chosen subset of the Banach space  $PC^1(J, X)$ . Obviously,  $X_r$  is a convex subset of  $PC^1(J, X)$ . To apply Schauder's fixed point theorem, we shall show that  $\mathcal{F}(X_r)$  is a precompact subset of  $X_r$ .

Let  $u \in X_r$ . Then from (3.2) by virtue of assumption **A3** we obtain

$$\begin{aligned}
\|\mathcal{F}u\|_{PC} & \leq \max_{t \in J} \|C(t)\| \cdot \|x_0\| + \max_{t \in J} \|S(t)\| (\|x_1\| + \|g(u)\|) \\
& + \max_{t \in J} \int_0^t \|S(t-s)\| \cdot \|f(s, u(s), u(b_1(s)), \dots, u(b_m(s)), \\
& \quad u'(s), u'(b_1(s)), \dots, u'(b_m(s)))\| ds \\
& + \max_{t \in J} \sum_{0 < \tau_k < t} \|C(t - \tau_k)\| \cdot \|I_k(u(\tau_k))\| \\
& + \max_{t \in J} \sum_{0 < \tau_k < t} \|S(t - \tau_k)\| \cdot \|\bar{I}_k(u(\tau_k), u'(\tau_k))\| \\
& \leq M [\|x_0\| + \|x_1\| + C_2 + TC_1 + \kappa(C_3 + C_4)].
\end{aligned}$$

Similarly, from (3.3) by virtue of assumption **A3** we obtain

$$\left\| \frac{d}{dt} \mathcal{F}u \right\|_{PC} \leq M [\|A\| \cdot \|x_0\| + \|x_1\| + C_2 + TC_1 + \kappa(\|A\|C_3 + C_4)],$$

and thus

$$\|\mathcal{F}u\|_{PC^1} = \|\mathcal{F}u\|_{PC} + \left\| \frac{d}{dt} \mathcal{F}u \right\|_{PC}$$



$$\leq M \{(1 + \|A\|)\|x_0\| + 2\|x_1\| + 2C_2 + 2TC_1 + \kappa[(1 + \|A\|)C_3 + 2C_4]\} \leq r.$$

The last inequality shows that  $\mathcal{F}u \in X_r$  for  $u \in X_r$ , that is,  $\mathcal{F}(X_r) \subset X_r$ .

It also implies that the family of functions  $\{\mathcal{F}u : u \in X_r\}$  is equibounded. Finally, we shall show that it is also equicontinuous on each interval of continuity  $J_k$ ,  $k = \overline{0, \kappa}$ .

Let  $u \in X_r$  and  $0 \leq t_1 < t_2 \leq T$ . Then we have

$$\begin{aligned} (\mathcal{F}u)(t_2) - (\mathcal{F}u)(t_1) &= (C(t_2) - C(t_1))x_0 + (S(t_2) - S(t_1))(x_1 - g(u)) \\ &+ \int_0^{t_1} (S(t_2 - s) - S(t_1 - s))f(s, u(s), u(b_1(s)), \dots, u(b_m(s)), \\ &\quad u'(s), u'(b_1(s)), \dots, u'(b_m(s)))ds \\ &+ \int_{t_1}^{t_2} S(t_2 - s)f(s, u(s), u(b_1(s)), \dots, u(b_m(s)), \\ &\quad u'(s), u'(b_1(s)), \dots, u'(b_m(s)))ds \\ &+ \sum_{0 < \tau_k < t_1} (C(t_2 - \tau_k) - C(t_1 - \tau_k))I_k(u(\tau_k)) + \sum_{t_1 \leq \tau_k < t_2} C(t_2 - \tau_k)I_k(u(\tau_k)) \\ &+ \sum_{0 < \tau_k < t_1} (S(t_2 - \tau_k) - S(t_1 - \tau_k))\bar{I}_k(u(\tau_k), u'(\tau_k)) \\ &+ \sum_{t_1 \leq \tau_k < t_2} S(t_2 - \tau_k)\bar{I}_k(u(\tau_k), u'(\tau_k)). \end{aligned}$$

From here by virtue of assumption **A3** we derive the inequality

$$\begin{aligned} &\|(\mathcal{F}u)(t_2) - (\mathcal{F}u)(t_1)\| \\ &\leq \|C(t_2) - C(t_1)\| \cdot \|x_0\| + \|S(t_2) - S(t_1)\|(\|x_1\| + C_2) \\ &+ \int_0^{t_1} \|S(t_2 - s) - S(t_1 - s)\| ds \cdot C_1 + (t_2 - t_1)MC_1 \\ &+ \sum_{0 < \tau_k < t_1} \|C(t_2 - \tau_k) - C(t_1 - \tau_k)\|C_3 + i(t_1, t_2)MC_3 \\ &+ \sum_{0 < \tau_k < t_1} \|S(t_2 - \tau_k) - S(t_1 - \tau_k)\|C_4 + i(t_1, t_2)MC_4, \end{aligned}$$

where  $i(t_1, t_2)$  is the number of instants of impulse effect in the interval  $[t_1, t_2]$ .

We first notice that the right-hand side of this inequality is independent of the choice of  $u \in X_r$ . Further on, due to the uniform continuity of the functions  $C(t)$  and  $S(t)$  on  $J$  in the operator norm, all norms in the right-hand side of this inequality can be made uniformly small for sufficiently small  $t_2 - t_1$ . Finally,  $i(t_1, t_2)$  is zero for  $t_1, t_2$  both in one of the intervals of continuity  $J_k$ ,  $k = \overline{0, \kappa}$ , and a positive integer otherwise. This shows that the family of functions  $\{\mathcal{F}u : u \in X_r\}$  is equicontinuous on each interval of continuity. Since all assumptions of Arzela-Ascoli's theorem are satisfied on each interval of continuity, the set  $\mathcal{F}(X_r)$  is a precompact subset of  $X_r$ . According to Schauder's fixed point theorem, the operator  $\mathcal{F}$  has a fixed point  $x$  in  $X_r$ , which is a mild solution of problem (2.1).  $\square$

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